Suboptimal Controller Synthesis for Cart-poles and Quadrotors via Sums-of-Squares

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Introduction

- Many interesting robotic tasks are naturally formulated as optimal control problems (OCP).
- Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation provide a sound theoretical framework to study the solution to OCPs.
- Sums-of-squares (SOS) optimization is a promising tool to synthesize certifiable controllers

Objectives

1. Find good **under- and over- approximations** of the value function using SOS.
2. Synthesize controllers with **bounded suboptimal performance** for various underactuated robotic systems.
3. Perform **regional analysis** on the closed-loop system driven by the controllers.

Problem Statement

- Continuous-time, nonlinear, control-affine systems

$$\dot{x}(t) = f(x(t), u(t)) = f_1(x(t)) + f_2(x(t))u(t),$$
$$u(t) \in \mathcal{U} = \{u | u_{\min} \le u \le u_{\max}\}$$

- HJB equation defines a necessary and sufficient condition for value function

$$\forall x,\ \min_{u\in\mathcal{U}} \left[l(x,u) + \frac{\partial J^*}{\partial x} f(x,u) \right] = 0$$

$$\forall x,\ \min_{u\in\mathcal{U}} \left[l(x,u) + \frac{\partial \underline{J}}{\partial x} f(x,u) \right] \ge 0$$

global under-approximation

$$\forall x,\ \min_{u\in\mathcal{U}} \left[l(x,u) + \frac{\partial \bar{J}}{\partial x} f(x,u) \right] \le 0$$

global over-approximation

- Closed-form optimal controller

$$\pi^*(x) = \text{clamp}\left(-\frac{1}{2} R^{-1} f_2(x)^T \frac{\partial J^T}{\partial x}, u_{\min}, u_{\max} \right)$$

Find tight local value function under- and over-approximations with low-degree polynomials.

Value Function Approximations

- **Local value function under-approximation**

integral metric to push up $\underline{J}$

value-function-like

regional HJB inequality satisfaction

$$\underline{J}^* := \operatorname*{argmax}_{\underline{J}(x)\ge 0} \int_{\mathcal{X}} \underline{J}(x)\, dx$$
$$\text{s.t.}\quad l(x,u) + \frac{\partial \underline{J}}{\partial x} f(x,u) \ge 0, \quad \forall x \in \mathcal{X}^h, \forall u \in \mathcal{U}$$

- **Suboptimal controller from under-approximation**

$$\underline{\pi}^*(x) = \text{clamp}\left(-\frac{1}{2} R^{-1} f_2(x)^T \frac{\partial \underline{J}^{*T}}{\partial x}, u_{\min}, u_{\max} \right)$$

- **Local value function over-approximation**

surrogate policy

$$\bar{J}^* := \operatorname*{argmin}_{\bar{J}(x)\ge 0} \int_{\mathcal{X}} \bar{J}(x)\, dx$$
$$\text{s.t.}\quad l(x,\bar{\pi}(x)) + \frac{\partial \bar{J}}{\partial x} f(x,\bar{\pi}(x)) \le 0, \quad \forall x \in \mathcal{X}^h, \forall \bar{\pi}(x) \in \mathcal{U}$$

- **Suboptimal controller from over-approximation**

$$\bar{\pi}^*(x) = \text{clamp}\left(-\frac{1}{2} R^{-1} f_2(x)^T \frac{\partial \bar{J}^{*T}}{\partial x}, u_{\min}, u_{\max} \right)$$

- Can be cast as SOS optimization programs

Regional Analysis

- Region of guaranteed controller performance (ROGCP): largest invariant set over which HJB inequalities are satisfied.
- Region of attraction (ROA): largest invariant set over which states can be stabilized with $\underline{\pi}^*$ ($\bar{\pi}^*$).
- $\bar{J}^*$ is a Lyapunov function whose sublevel sets can inner approximate both ROGCP and ROA using SOS programs.

$$\mathcal{C}_{\rho^*} := \{x | \bar{J}^*(x) < \rho^*\}$$

desired set point

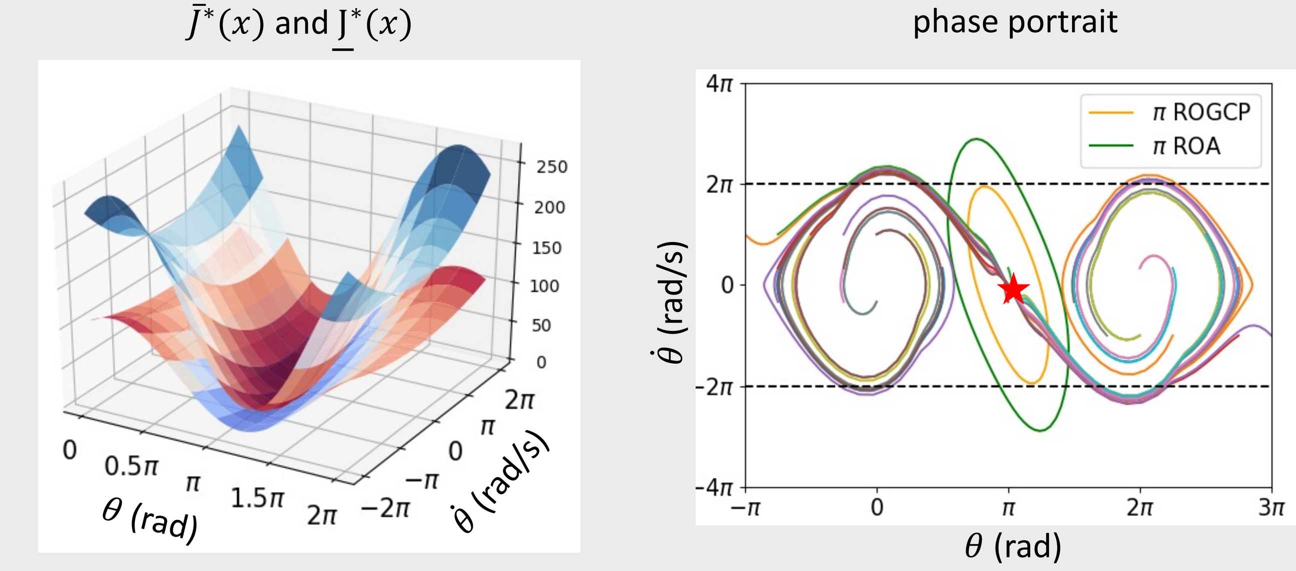
$$\rho^* := \max_{\rho, \lambda(x)} \rho$$
$$\text{s.t.}\quad ((x - x_d)^T (x - x_d))^d (\bar{J}^*(x) - \rho)$$
$$+ \lambda(x) \frac{\partial \bar{J}^*}{\partial x} f(x, \bar{\pi}(x)) \ge 0,\ \forall \bar{\pi}(x) \in \mathcal{U}$$

	HJB inequality satisfied	Guaranteed controller properties
ROGCP	yes	suboptimality + stability
ROA	not guaranteed	stability

Experimental Results

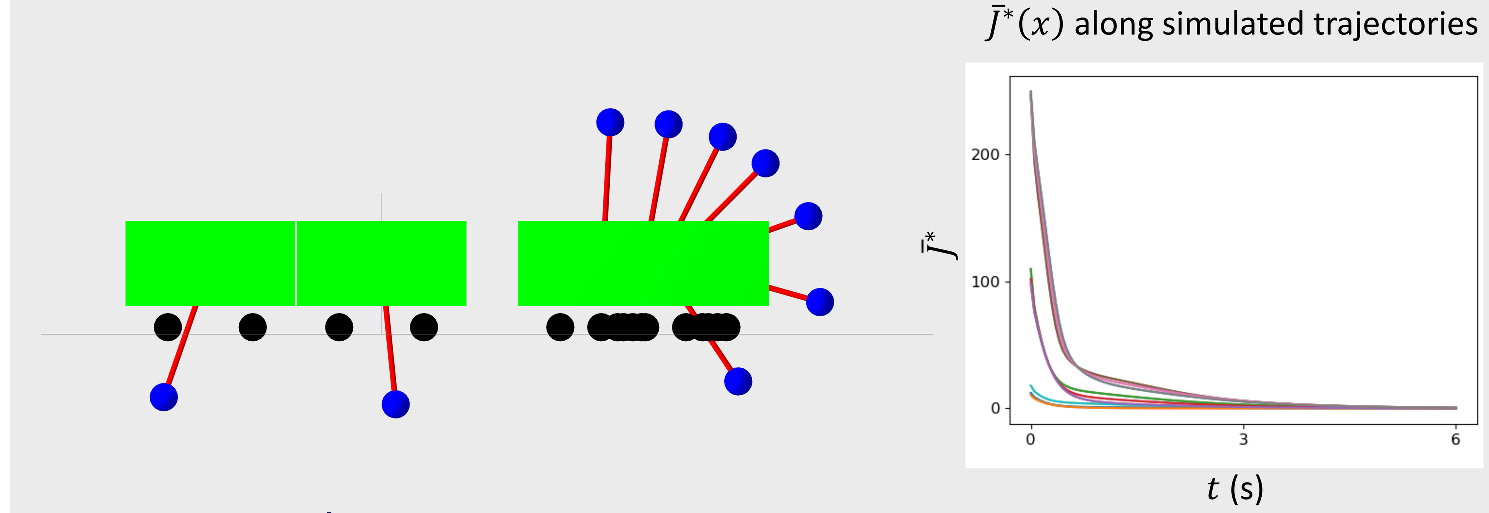
- Inverted Pendulum

Challenging torque limit ⇒ nontrivial pumping



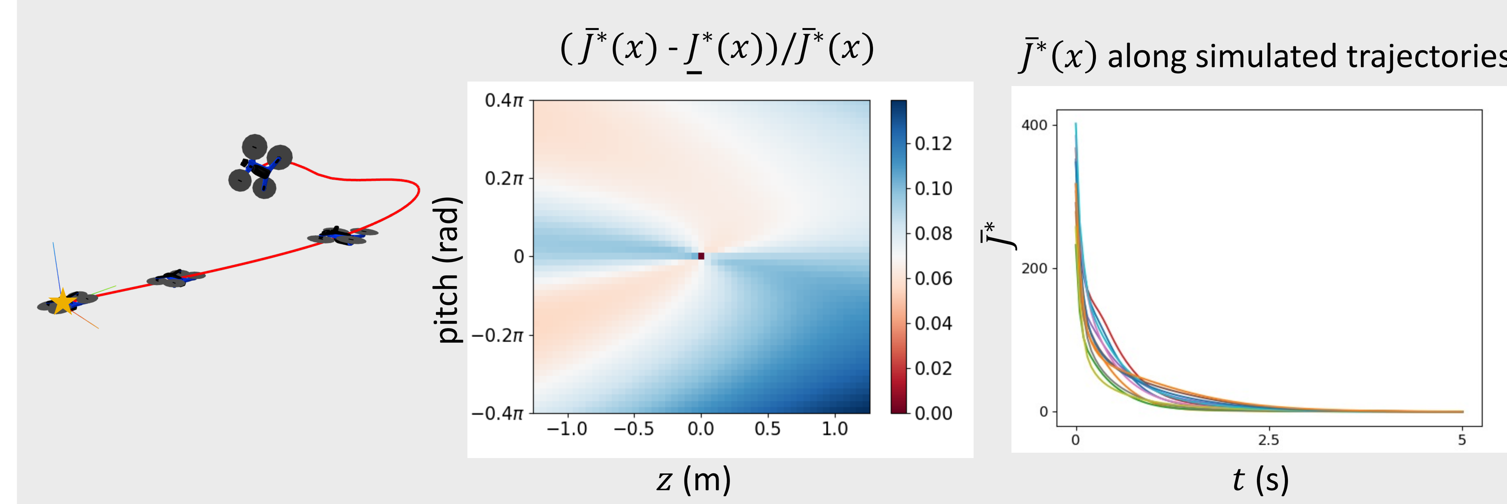
- Cartpole swing-up

Challenging torque limit, chaotic dynamics ⇒ nontrivial pumping



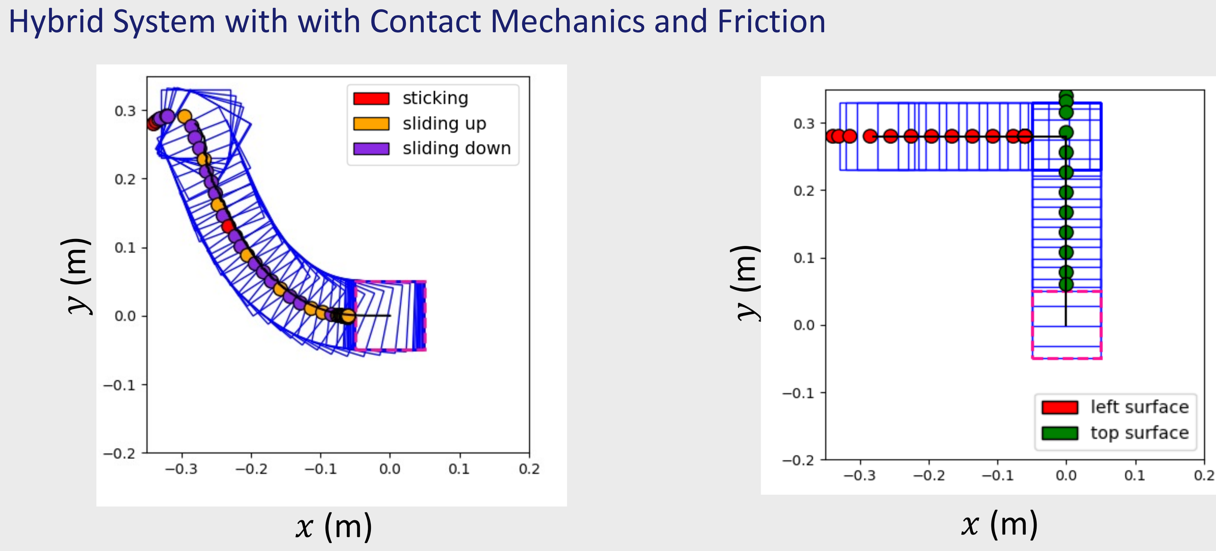
- 3D Quadrotor

13 dimension, underactuated, large ROA



- Planar Pusher

Hybrid System with with Contact Mechanics and Friction



- **Takeaway**: first SOS-based time-invariant controller that can swing up and stabilize a cart-pole, and push the planar slider to the desired pose